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Modified Seiberg–Witten Monopole Equations and Their Exact Solutions

Zhong-Qi Ma,[1] Bo-Yuan Hou,[2] and Fu-Zhong Yang[2]

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The modified Seiberg–Witten monopole equations are presented. The equations have analytic solutions in the whole 1 + 3 space with finite energy. The physical meaning of the equations and solutions are discussed.

1. INTRODUCTION

Witten (1994) developed an elegant dual approach that greatly simplified the Donaldson (1983, 1993) theory of four-manifolds. This approach starts from the Seiberg–Witten monopole equations (Seiberg and Witten, 1994)

$$\sum_{\mu=1}^{4} \gamma_\mu D_\mu \psi(x) = \sum_{\mu=1}^{4} \gamma_\mu (\partial_\mu - ieW_\mu)\psi(x) = 0 \tag{1}$$

$$F^{+}_{\mu\nu}(x) = -\frac{i}{2}\,\bar{\psi}(x)\gamma^{+}_{\mu\nu}\psi(x) \tag{2}$$

where x denotes the coordinates in the 1 + 3 space, and the natural units $\hbar = c = 1$ are employed.

Witten noted that in flat space the Seiberg–Witten equations admit no L^2 solutions. Seiberg–Witten monopole equations were directly generalized to the non-Abelian case on four-manifolds (Labastida and Mariño, 1995), which also admits no L^2 solutions in the flat space. However, Freund (1995) found a simple *non-*L^2 solution, based on a fermion moving in a U(1) Dirac monopole field (Dirac, 1948). This solution has a singularity at the origin, and a singular string in the monopole can be removed by the concept of a section (Wu and Yang, 1975).

[1] Institute of High Energy Physics, P.O. Box 918(4), Beijing 100039, China.
[2] Graduate School, University of Science and Technology of China, Beijing 100039, China.

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It is reasonable to extend Freund's solution by considering the solution based on a fermion moving in the smooth 't Hooft–Polyakov monopole field ('t Hooft, 1974; Polyakov, 1974; Prasad and Sommerfield, 1975). This solution is analytic in the whole space with finite energy such that it does not satisfy the Seiberg–Witten equations (Seiberg and Witten, 1994) and their direct generalization (Labastida and Mariño, 1995), but the modified ones. In other words, the Seiberg–Witten equations has to be modified to be compatible with the SU(2) monopole solution.

As pointed out by Freund (1995), the spinor in his solution satisfies the Weyl–Dirac equation, and the spinor field constructs the Coulomb field. It is our starting point that these important properties should be kept in the modified equations and their solutions.

In this paper we modify the Seiberg–Witten equation (2), which is quadratic in the spinor field, to be a linear one. Surprisingly, we find that the modified Seiberg–Witten equations have an exact solution, analytic in the whole 1 + 3 space with finite energy.

This paper is organized as follows. In Section 2 we review the analytic SU(2) monopole solution, and reexpress the self-dual solution as a two-component spinor. In Section 3, due to zero mass, the Weyl–Dirac equation can be separated into the two-component spinor form. We sketch the calculation for the two-component spinor field moving in an analytic, self-dual, static, and spherically symmetric SU(2) monopole field without external source, and find that the solution is nothing but the self-dual monopole solution in the spinor form. Thus, we modify the Seiberg–Witten equation (2) to be linear in the spinor field. The physical meaning of the equations and solutions is discussed in Section 4.

2. SELF-DUAL SU(2) MONOPOLE SOLUTION

Prasad and Sommerfield (1975) found the analytic SU(2) monopole solution with finite energy. If one changes the Higgs field to the fourth component of the gauge potential, an analytic, self-dual, static, and spherically symmetric SU (2) monopole solution without external source can be obtained uniquely (Hou and Hou, 1981):

$$\mathbf{W} = \frac{\hat{r} \wedge \mathbf{T}}{er}(1 - r\phi(r)), \qquad W_4 = \frac{iG(r)}{e}\mathbf{T} \cdot \hat{r}$$

$$\phi(r) = \frac{\beta}{\sinh(\beta r)} \sim \begin{cases} r^{-1} & \text{when} \quad r \to 0 \\ O(e^{-\beta r}) & \text{when} \quad r \to \infty \end{cases} \tag{3}$$

$$iG(r) = r^{-1} - \beta \coth(\beta r) \sim \begin{cases} O(r) & \text{when} \quad r \to 0 \\ -\beta & \text{when} \quad r \to \infty \end{cases}$$

where e is the unit electric charge, $\hat{r}$ is the unit radial vector, β is a constant with positive real part, and **T** are the SU(2) generators in the isospin space. An anti-self-dual solution can be obtained by replacing $G(r)$ with $-G(r)$.

The gauge field

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ie(W_\mu W_\nu - W_\nu W_\mu) \tag{4}$$

is self-dual such that the magnetic field $B_k = \Sigma_{ij}\, \epsilon_{ijk} G_{ij}/2$ is equal to the electric field $-iE_k = G_{k4}$:

$$\mathbf{B}(x) = -i\mathbf{E}(x) = B_\parallel(r)\hat{r}(\mathbf{T}\cdot\hat{r}) + B_\perp(r)(\mathbf{T} - \hat{r}(\mathbf{T}\cdot\hat{r}))$$

$$B_\parallel(r) = -\frac{1}{er^2} + \frac{\beta^2}{e\sinh^2(\beta r)} \sim \begin{cases} -\beta^2/(3e^2) & \text{when} \quad r \to 0 \\ -1/(er^2) & \text{when} \quad r \to \infty \end{cases} \tag{5}$$

$$B_\perp(r) = \frac{\beta}{er\sinh(\beta r)} - \frac{\beta^2\coth(\beta r)}{e\sinh(\beta r)} \sim \begin{cases} -\beta^2/(3e^2) & \text{when} \quad r \to 0 \\ O(e^{-\beta r}) & \text{when} \quad r \to \infty \end{cases}$$

Note that $B_\parallel(r)$ and $B_\perp(r)$ tend to the same constant when r goes to zero, so that they cancel each other and **B** is single-valued at the origin.

Usually, there are two ways to express the gauge field **B**. One expression is given in (5), where the generator **T** is included in the expression. Another expression does not include the generator **T**, and the components $\mathbf{B}_\rho$ are written as a 3×1 column matrix. Choosing the spherical harmonic bases for **T**,

$$T_1 = -2^{-1/2}(T_x + iT_y), \qquad T_0 = T_z, \qquad T_{-1} = 2^{-1/2}(T_x - iT_y) \tag{6}$$

we have

$$\mathbf{B}(x) = -i\mathbf{E}(x) = B_\parallel(r)\hat{r}(\mathbf{t}\cdot\hat{r}) + B_\perp(r)[\mathbf{t} - \hat{r}(\mathbf{t}\cdot\hat{r})] \tag{7}$$

$$\mathbf{t}_x = \begin{pmatrix} -2^{-1/2} \\ 0 \\ 2^{-1/2} \end{pmatrix}, \qquad \mathbf{t}_y = \begin{pmatrix} i2^{-1/2} \\ 0 \\ i2^{-1/2} \end{pmatrix}, \qquad \mathbf{t}_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

It is easy to check that

$$T_a t_b = i \sum_{c=x,y,z} \epsilon_{abc} t_c \tag{8}$$

where T_x, T_y, and T_z are the generators in the representation $D^1(SU(2))$.

The self-dual monopole solution $\mathbf{B}(x)$ transforms according to the representation $D^{10}(A)$ of the Lorentz group, where A is any Lorentz transformation. As usually done in the twistor theory (Jackiw and Rebbi, 1996; Brown *et al.*, 1978; Mottola, 1978; Penrose and Rindler, 1986), the solution can be rewritten as a matrix form $\bar{\sigma} \cdot \mathbf{B}(x)$, which transforms in the Lorentz transformation A as follows:

$$\bar{\sigma} \cdot \mathbf{B}(x) \to D^{(1/2)0}(A)\{\bar{\sigma} \cdot \mathbf{B}(A^{-1}x)\}D^{(1/2)0}(A)^{-1} \tag{9}$$

where $\bar{\sigma}$ is the Pauli matrix.

Multiplying $\bar{\sigma} \cdot \mathbf{B}(x)$ on a unit spinor $\chi(m)_b$, which transforms in the Lorentz transformation A as

$$\chi(m)_a \to \sum_{b=\pm 1/2} D_{ab}^{(1/2)0}(A)\chi(m)_b \tag{10}$$

$$\chi(1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi(-1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

we obtain two new spinors $G^{+}(x, m)$, $m = \pm 1/2$, related to the self-dual monopole solution $\mathbf{B}(r)$:

$$\begin{aligned} G^{+}(x, m)_a^{\tau} &\equiv \sum_{b=\pm 1/2} \{\bar{\sigma} \cdot \mathbf{B}(x)\}_{ab}^{\tau}\chi(m)_b \\ &= B_{\parallel}(r)(\bar{\sigma} \cdot \hat{r})_{am}(\mathbf{t} \cdot \hat{r})_{\tau} + B_{\perp}(r)\{(\bar{\sigma} \cdot \theta)_{am}(\mathbf{t} \cdot \theta)_{\tau} + (\bar{\sigma} \cdot \hat{\varphi})_{am}(\mathbf{t} \cdot \hat{\varphi})_{\tau}\} \end{aligned} \tag{12}$$

where a and m run over $\pm 1/2$, the isospinor index $\tau = 1, 0, -1$, and $\hat{r}$, θ, and $\hat{\varphi}$ are the three unit vectors in the spherical coordinate system. We have

$$\begin{aligned} (\bar{\sigma} \cdot \hat{r}) &= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}, \qquad (\mathbf{t} \cdot \hat{r}) = \begin{pmatrix} -2^{-1/2}\sin\theta\, e^{-i\varphi} \\ \cos\theta \\ 2^{-1/2}\sin\theta\, e^{i\varphi} \end{pmatrix} \\ (\bar{\sigma} \cdot \theta) &= \begin{pmatrix} -\sin\theta & \cos\theta\, e^{-i\varphi} \\ \cos\theta\, e^{i\varphi} & \sin\theta \end{pmatrix}, \qquad (\mathbf{t} \cdot \theta) = \begin{pmatrix} -2^{-1/2}\cos\theta\, e^{-i\varphi} \\ -\sin\theta \\ 2^{-1/2}\cos\theta\, e^{i\varphi} \end{pmatrix} \\ (\bar{\sigma} \cdot \hat{\varphi}) &= \begin{pmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{pmatrix}, \qquad (\mathbf{t} \cdot \hat{\varphi}) = \begin{pmatrix} i2^{-1/2}e^{-i\varphi} \\ 0 \\ i2^{-1/2}e^{i\varphi} \end{pmatrix} \end{aligned} \tag{13}$$

$G^{+}(x, m)$ is a direct product of a 2×1 spinor matrix and a 3×1 isospinor matrix, which transforms in the Lorentz transformation like a spinor:

$$G^{+}(x, m)_a^{\tau} \to \sum_{b=\pm 1/2} D_{ab}^{(1/2)0}(A)G^{+}(A^{-1}x, m)_b^{\tau} \tag{14}$$

Note that two spinors $G^{+}(x, m)$ satisfy a complex conjugate relation:

$$G^{+}(x, -1/2)^{\tau}_{a} = \sum_{b=\pm 1/2} \sum_{\lambda=-1}^{1} d^{1/2}_{ab}(\pi) d^{1}_{\tau\lambda}(\pi) \{G^{+}(x, 1/2)^{\lambda}_{b}\}^{*},$$
$$d^{i}_{ab}(\pi) = (-1)^{i+a} \delta_{a(-b)} \tag{15}$$

where the combinative coefficients have to be included due to the similarity transformations $d^{j}(\pi)$ between two equivalent reprepresentations $D^{j}(SU(2))$ and $D^{j}(SU(2))^{*}$.

Similarly, the anti-self-dual solution $G^{-}(x, m)$ transforms in the Lorentz transformation:

$$G^{-}(x, m)^{\tau}_{a} \rightarrow \sum_{b=\pm 1/2} D^{0(1/2)}_{ab}(A) G^{-}(A^{-1}x, m)^{\tau}_{b} \tag{16}$$

3. SPINOR SOLUTION

Let $\phi(x)$ be a spinor field with spin 1/2 and isospin 1, satisfying the Weyl–Dirac equation (1). By making use of the γ matrices

$$\bar{\bar{\gamma}} = \begin{pmatrix} 0 & i\bar{\bar{\sigma}} \\ -i\bar{\bar{\sigma}} & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

we can introduce the two-component spinor fields with isospin one:

$$u^{\pm}(x) \sim \frac{1}{2}(1 \pm \gamma_5)\psi(x), \qquad \psi(x) = \begin{pmatrix} u^{+}(x) \\ u^{-}(x) \end{pmatrix} \tag{17}$$

satisfying

$$\pm i\bar{\bar{\sigma}} \cdot (\nabla - ie\mathbf{W}) u^{\pm}(x) = i\frac{\partial}{\partial t} u^{\pm}(x) + ieW_4 u^{\pm}(x) \tag{18}$$

It is due to zero mass that the Weyl–Dirac equation can be separated into the two-component form. Since

$$\gamma_1\gamma_2 = \gamma_5\gamma_3\gamma_4 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \gamma_3\gamma_4$$

$u^{\pm}(x)$ in (18) is the two-component spinor belonging to $D^{(1/2)0}(A)$ and $D^{0(1/2)}(A)$, respectively:

$$u^{+}(x)^{\tau}_{a} \rightarrow \sum_{b=\pm 1/2} D^{(1/2)0}_{ab}(A) u^{+}(A^{-1}x)^{\tau}_{b} \tag{19}$$
$$u^{-}(x)^{\tau}_{a} \rightarrow \sum_{b=\pm 1/2} D^{0(1/2)}_{ab}(A) u^{-}(A^{-1}x)^{\tau}_{b}$$

where $a = \pm 1/2$ is the spinor index and $\tau = \pm 1$ and 0 is the isospinor index.

Twenty years ago the zero-energy spinor solutions with isospin 1 of the Weyl–Dirac equation (1) were obtained (Mottola, 1978; Hou and Hou, 1979; Rossi, 1982), where the spinor field moves in an analytic, self-dual, static, and spherically symmetric SU(2) monopole field without external source. We sketch the calculation as follows.

Equation (18) is spherically symmetric, so that the general angular momentum $\mathbf{J}$ is conserved:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} + \mathbf{T}, \qquad \mathbf{S} = \bar{\sigma}/2 \tag{20}$$

The two-component spinors $u^{\pm}_{jm}(x)$ can be expanded with respect to the common eigenfunctions of J^2, J_z, S^2, $S \cdot \hat{r}$, T^2, and $T \cdot \hat{r}$:

$$u^{\pm}_{jm}(x)^{\tau}_{a} = \sum_{b=\pm 1/2} \sum_{\lambda=-1}^{1} f^{\pm}_{jmb\lambda}(r) \eta^{j}_{mb\lambda}(\hat{r})^{\tau}_{a} e^{-iEt} \tag{21}$$

$$\eta^{j}_{mb\lambda}(\hat{r})^{\tau}_{a} = \left(\frac{2j+1}{4\pi}\right)^{1/2} D^{j}_{m(b+\lambda)}(\varphi, \theta, 0)^{*} D^{1/2}_{ab}(\varphi, \theta, 0) D^{1}_{\tau\lambda}(\varphi, \theta, 0) \tag{22}$$

where θ and φ are the polar and azimuthal angles of $\hat{r}$, the spinor subscript a runs over $\pm 1/2$, and the isospinor subscript τ runs over 1, 0, and -1. Expressing the spinor and the isospinor as column matrixes, respectively, we can rewrite $\eta^{j}_{mb\lambda}(\hat{r})$ as a direct product of a 2×1 spinor matrix and a 3×1 isospinor matrix, satisfying

$$\begin{aligned} J^2 \eta^{j}_{mb\lambda}(\hat{r}) &= j(j+1)\eta^{j}_{mb\lambda}(\hat{r}), & S^2 \eta^{j}_{mb\lambda}(\hat{r}) &= (3/4)\eta^{j}_{mb\lambda}(\hat{r}) \\ T^2 \eta^{j}_{mb\lambda}(\hat{r}) &= 2\eta^{j}_{mb\lambda}(\hat{r}), & J_z \eta^{j}_{mb\lambda}(\hat{r}) &= m\eta^{j}_{mb\lambda}(\hat{r}) \\ (S \cdot \hat{r})\eta^{j}_{mb\lambda}(\hat{r}) &= b\eta^{j}_{mb\lambda}(\hat{r}), & (T \cdot \hat{r})\eta^{j}_{mb\lambda}(\hat{r}) &= \lambda\eta^{j}_{mb\lambda}(\hat{r}) \end{aligned} \tag{23}$$

It is easy to show the following formulas:

$$\begin{aligned} \bar{\sigma} \cdot \bar{D} &= (\bar{\sigma} \cdot \hat{r})\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - r^{-1}(\bar{\sigma} \cdot \hat{r})K \\ &\quad + i\phi(r)\bar{\sigma} \cdot (\hat{r} \wedge \mathbf{T}) \\ K &= \bar{\sigma} \cdot \{\mathbf{r} \wedge (-i\nabla - \mathbf{r} \wedge \mathbf{T}/r^2)\} + 1 \\ \bar{\sigma} \cdot (\hat{r} \wedge \mathbf{T})\eta^{j}_{mb\lambda}(\hat{r}) &= i2bA_{b\lambda}\eta^{j}_{m(-b)(\lambda+2b)}(\hat{r}) \\ A_{b\lambda} &= \{(9/4) - (b+\lambda)^2\}^{1/2} \\ K\eta^{j}_{mb\lambda}(\hat{r}) &= K_{\lambda}\eta^{j}_{m(-b)\lambda}(\hat{r}) \\ K_{\lambda} &= \{(j+1/2)^2 - \lambda^2\}^{1/2} \end{aligned} \tag{24}$$

Hence, equation (18) for $E = 0$ becomes

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) f^{\pm}_{jmb\lambda}(r) - K_{\lambda} r^{-1} f^{\pm}_{jm(-b)\lambda}(r) + A_{b\lambda}\phi(r) f^{\pm}_{jm(-b)(\lambda+2b)}(r) = \mp iG(r) 2b\lambda f^{\pm}_{jmb\lambda}(r) \tag{25}$$

From the convergent condition, the only analytical solutions were obtained for $j = 1/2$ (Hou and Hou, 1979):

$$\begin{aligned} f^{-}_{jmb\lambda}(r) &= 0 \\ f^{+}_{(1/2)m(1/2)0}(r) &= -f^{+}_{(1/2)m(-1/2)0}(r) = c_m \sqrt{2\pi} B_{\|}(r) \\ f^{+}_{(1/2)m(1/2)(-1)}(r) &= -f^{+}_{(1/2)m(-1/2)1}(r) = c_m 2\sqrt{\pi} B_{\perp}(r) \end{aligned} \tag{26}$$

where c_m is an arbitrary constant due to the linear property of (18), and $B_{\|}$ and $B_{\perp}$ were given in (5). We choose the constant $c_m = 1$ for convenience. The remaining components $f^{+}_{jmb\lambda}(r)$ are vanishing.

In terms of the exact form of $D^{j}_{mm'}(\alpha, \beta, \gamma)$ (Rose 1957),

$$D^{j}_{mm'}(\alpha, \beta, \gamma) = \sum_{n} \frac{(-1)^n \{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!\}^{1/2}}{(j+m-n)!\,(j-m'-n)!\,n!\,(n-m+m')!} \times e^{-im\alpha}[\cos(\beta/2)]^{2j+m-m'-2n}[\sin(\beta/2)]^{2n-m+m'} e^{-im'\gamma} \tag{27}$$

we have

$$\eta^{1/2}_{(1/2)(1/2)0}(\hat{r}) - \eta^{1/2}_{(1/2)(-1/2)0}(\hat{r}) = (2\pi)^{-1/2} \begin{pmatrix} \cos\theta \\ \sin\theta\, e^{i\varphi} \end{pmatrix} \otimes (\mathbf{t}\cdot\hat{r})$$

$$\eta^{1/2}_{(-1/2)(1/2)0}(\hat{r}) - \eta^{1/2}_{(-1/2)(-1/2)0}(\hat{r}) = (2\pi)^{-1/2} \begin{pmatrix} \sin\theta\, e^{-i\varphi} \\ -\cos\theta \end{pmatrix} \otimes (\mathbf{t}\cdot\hat{r})$$

$$\eta^{1/2}_{(1/2)(1/2)(-1)}(\hat{r}) - \eta^{1/2}_{(1/2)(-1/2)1}(\hat{r}) = (4\pi)^{-1/2} \left\{ \begin{pmatrix} -\sin\theta \\ \cos\theta\, e^{i\varphi} \end{pmatrix} \otimes (\mathbf{t}\cdot\theta) + (4\pi)^{-1/2} \begin{pmatrix} 0 \\ ie^{i\varphi} \end{pmatrix} \otimes (\mathbf{t}\cdot\hat{\varphi}) \right\} \tag{28}$$

$$\eta^{1/2}_{(-1/2)(1/2)(-1)}(\hat{r}) - \eta^{1/2}_{(-1/2)(-1/2)1}(\hat{r}) = (4\pi)^{-1/2} \left\{ \begin{pmatrix} \cos\theta\, e^{-i\varphi} \\ \sin\theta \end{pmatrix} \otimes (\mathbf{t}\cdot\theta) + (4\pi)^{-1/2} \begin{pmatrix} -ie^{-i\varphi} \\ 0 \end{pmatrix} \otimes (\mathbf{t}\cdot\hat{\varphi}) \right\}$$

Thus, $u^{-}_{jm}(x) = 0$, and $u^{+}_{(1/2)(1/2)}(x)$ and $u^{+}_{(1/2)(-1/2)}(x)$ satisfy a complex conjugate relation like (13):

$$u^{+}_{(1/2)(-1/2)}(x)^{\tau}_{a} = \sum_{c=\pm 1/2}\sum_{\lambda=-1}^{1} d^{1/2}_{ac}(\pi)\, d^{1}_{\tau\lambda}(\pi)(u^{+}_{(1/2)(1/2)}(x)^{\lambda}_{c})^{*} \tag{29}$$

Now, it is evident that the solution $u^{+}_{(1/2)m}(x)^{\tau}_{a}$ is nothing but the self-dual monopole solution $G^{+}(x, m)^{\tau}_{a}$ in the spinor form:

$$u^{+}_{(1/2)m}(x)^{\tau}_{a} = G^{+}(x, m)^{\tau}_{a} = \sum_{b=\pm 1/2} \{\bar{\sigma} \cdot \mathbf{B}(x)\}^{\tau}_{ab}\chi(m)_{b} \tag{30}$$

where m and $a = \pm 1/2$, and $\tau = 1, 0, -1$. A similar formula was obtained from the supersymmetric theory (Osborn, 1979).

Now, we call (1) [or its equivalent form (18)] and (30) the modified Seiberg–Witten monopole equations. The main change from (2) to (30) is that the bilinear form of ψ in (2) becomes the linear form in (30). The gauge group changes from an Abelian group to a non-Abelian one. The great merit of this change is that the modified equations have analytic solutions in the whole $1 + 3$ space with finite energy. Comparing (19) with (14), we conclude that (30) is also Lorentz covariant.

If we replace the analytic, self-dual, static, and spherically symmetric SU(2) monopole solution without external source by the anti-self-dual one, (30) change to

$$u^{-}_{(1/2)m}(x)^{\tau}_{a} = G^{-}(x, m)^{\tau}_{a} \tag{31}$$

4. CONCLUSION AND DISCUSSION

In summary, we have modified the Seiberg–Witten equations as

$$\begin{aligned} &\sum_{\mu=1}^{4} \gamma_{\mu}(\partial_{\mu} - ieW_{\mu})\psi(x) = 0 \\ &u^{+}_{(1/2)m}(x)^{\tau}_{a} = G^{+}(x, m)^{\tau}_{a} = \sum_{b=\pm 1/2} \{\bar{\sigma} \cdot \mathbf{B}(x)\}^{\tau}_{ab}\chi(m)_{b} \end{aligned} \tag{32}$$

The first equation describes a spinor field with isospin one moving in an analytic, self-dual, static, and spherically symmetric SU(2) monopole field without external source. Due to zero mass it can be separated into the two-component form. The second equation shows that the two-component spinor field relates with the self-dual gauge field in the spinor form directly. This is why the gauge field still satisfies the Yang–Mills equation without external source.

Since the gauge field is a hedgehog solution, its integral on a closed spherical face is

$$\frac{e}{4\pi}\frac{3}{I(I+1)(2I+1)}\,Tr\left\{\iint \mathbf{B}(\mathbf{T}\cdot\hat{r})\cdot d\mathbf{S}\right\} = er^2B_{\|}(r) \sim -1$$
$$\text{when} \quad r \to \infty \tag{33}$$

where I denotes the isospin in the space on which $\mathbf{T}$ acts. This is the first Chern number of the gauge field, and the asymptotic form shows that the total magnetic charge is $-1/e$. Furthermore,

$$(4\pi)^{-1}Tr\left\{\iint \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma}G_{\mu\nu}G_{\rho\sigma}\, dS\right\} = 2r^2\{B_{\|}(r)^2 + 2B_{\perp}(r)^2\} \sim O(r^{-2})$$
$$\text{at} \quad r \sim \infty \tag{34}$$

On the other hand, the spinor field $\psi_{jm}(x)$ with $j = 1/2$ satisfies

$$(4\pi r^2)^{-1}\int \psi_{jm}(x)^{\dagger}(\bar{\sigma}\cdot\hat{r})\psi_{jm}(x)\, dS$$
$$= (4\pi r^2)^{-1}\int \psi_{jm}(x)^{\dagger}(\mathbf{T}\cdot\hat{r})\psi_{jm}(x)\, dS = 0 \tag{35}$$

$$(4\pi)^{-1}\int \psi_{jm}(x)^{\dagger}\psi_{jm}(x)\, dS = (4\pi r^2)[B_{\|}(r)^2 + 2B_{\perp}(r)^2] \tag{36}$$

Therefore,

$$(16\pi^2)^{-1}\,Tr\left\{\iint \sum_{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma}G_{\mu\nu}G_{\rho\sigma}\, dS\right\}$$
$$= (32\pi^3)^{-1}\int \psi_{jm}(x)^{\dagger}\psi_{jm}(x)\, dS \tag{37}$$

where the integrand on the left hand side of (37) is nothing but the second Chern class.

The physical meaning of the modified Seiberg–Witten monopole equations should be further explored, and new solutions will be sought later. It will be discussed elsewhere whether or not the modified Seiberg–Witten equations are conformal covariant.

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